

REPRESENTABILITY OF COHOMOLOGICAL FUNCTORS OVER EXTENSION FIELDS

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ABSTRACT. We generalize a result of Orlov and Van den Bergh on the representability of a cohomological functor $H : D_{\text{Coh}}^b(X) \rightarrow \underline{\text{mod}}_L$ to the case where L is a field extension of the base field k of the variety X , with $\text{trdeg}_k L \leq 1$ or L purely transcendental of degree 2.

This result can be applied to investigate the behavior of an exact functor $F : D_{\text{Coh}}^b(X) \rightarrow D_{\text{Coh}}^b(Y)$ with X and Y smooth projective varieties and $\dim Y \leq 1$ or Y a rational surface. We show that for any such F there exists a “generic kernel” A in $D_{\text{Coh}}^b(X \times Y)$, such that F is isomorphic to the Fourier-Mukai transform with kernel A after composing both with the pullback to the generic point of Y .

1. INTRODUCTION

Let X be a smooth projective variety over an algebraically closed field k . In this paper we will generalize a result of Orlov and Van den Bergh [CKN01, Lemma 2.14] on the representability of a functor $H : D_{\text{Coh}}^b(X) \rightarrow \underline{\text{mod}}_k$ to the case of an extension field $k \subset L$:

Theorem 1.1. *Let X be a smooth projective variety over a field k . Let L be a finitely generated separable field extension of k with $\text{trdeg}_k L \leq 1$, or a purely transcendental field extension of transcendence degree 2 over k . Consider a contravariant, cohomological, finite type functor*

$$H : D_{\text{Coh}}^b(X) \rightarrow \underline{\text{mod}}_L$$

Then H is representable by an object $E \in D_{\text{Coh}}^b(X_L)$, i.e. there exists E such that for every $C \in D_{\text{Coh}}^b(X)$ we have

$$H(C) = \text{Mor}_{D_{\text{Coh}}^b(X_L)}(j^*C, E)$$

where $j^ : X_L \rightarrow X$ is the base change morphism.*

An interesting example of a functor as in Theorem 1.1 can be obtained from an exact functor $F : D_{\text{Coh}}^b(X) \rightarrow D_{\text{Coh}}^b(Y)$ between the bounded derived categories of two smooth projective varieties X and Y , where $\dim Y \leq 1$ or Y is a rational surface. To produce a functor as in the above theorem, we compose F with the pullback to the generic point of Y , take cohomology, then dualize to get a contravariant functor:

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$$\begin{array}{ccccccc}
D_{\text{Coh}}^b(X) & \xrightarrow{F} & D_{\text{Coh}}^b(Y) & \xrightarrow{i^*} & D_{\text{Coh}}^b(\eta) & \xrightarrow{H^0} & \underline{\text{mod}}_{K(Y)} \xrightarrow{D} \underline{\text{mod}}_{K(Y)} \\
& & & & & & \uparrow \\
& & & & & & H
\end{array}$$

Theorem 1.1 will thus allow us to tackle the question of whether a functor between the bounded derived categories of two smooth projective varieties is representable by a Fourier-Mukai transform. When $\dim Y \leq 1$ or Y is a rational surface we can answer positively to the question above after restricting to the generic point of Y :

Theorem 1.2. *Let X, Y be smooth projective varieties over a field k , where $\dim Y \leq 1$ or Y is a rational surface. Consider a covariant exact functor*

$$F : D_{\text{Coh}}^b(X) \rightarrow D_{\text{Coh}}^b(Y)$$

let $i : \eta \rightarrow Y$ the inclusion of the generic point of Y . Then there exists an object $A \in D_{\text{Coh}}^b(X \times Y)$ such that

$$i^* \circ F = i^* \circ \Phi_A,$$

where $\Phi_A(\cdot) := Rp_{2}(A \otimes^L Lp_1^*(\cdot))$ is the Fourier-Mukai transform with kernel A and $p_1 : X \times Y \rightarrow X, p_2 : X \times Y \rightarrow Y$ are the projection morphisms.*

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2. THE BASE CHANGE CATEGORY

In what follows, an abelian category \mathcal{A} does not automatically have any limits or colimits apart from the finite ones.

Given a field K , we will denote with $\underline{\text{mod}}_K$ the category of finite dimensional K -vector spaces, whereas $\underline{\text{Mod}}_K$ will denote the category of possibly infinite-dimensional K -vector spaces. $D(\mathcal{A})$ will denote the derived category of an abelian category \mathcal{A} .

Given an R -linear abelian category \mathcal{A} and an inclusion of rings $R \hookrightarrow S$, we can define the base change category \mathcal{A}_S as in [LVdB06, §4]:

Definition 2.1. *The category \mathcal{A}_S is given by pairs (C, ρ_C) where $C \in \text{Ob}(\mathcal{A})$ and $\rho_C : S \rightarrow \text{Hom}_{\mathcal{A}}(C, C)$ is an R -algebra map such that the composition $R \rightarrow S \rightarrow \text{Hom}_{\mathcal{A}}(C, C)$ gives back the R -algebra structure on \mathcal{A} . The morphisms in \mathcal{A}_S are the morphisms in \mathcal{A} compatible with the S -structure.*

Definition 2.2. *For each element $C \in \mathcal{A}$, the functor*

$$C \otimes_R - : \underline{\text{mod}}(R) \rightarrow \mathcal{A}$$

is the unique finite colimit preserving functor with $C \otimes R = C$.

This gives for each finitely presented R -algebra S a functor

$$- \otimes S : \mathcal{A} \rightarrow \mathcal{A}_S$$

to the base change category \mathcal{A}_S .

Proposition 2.3. [LVdB06, Proposition 4.3] *The functor $- \otimes S$ is left adjoint to the forgetful functor*

$$\begin{aligned} \text{Forget} : \mathcal{A}_S &\rightarrow \mathcal{A} \\ (C, \rho_C) &\mapsto C \end{aligned}$$

Whenever the context is clear, given an object $B \in \mathcal{A}_S$, we will still denote by B the corresponding object of \mathcal{A} obtained via the forgetful functor.

For the purposes of this discussion we will need a more general setting for base change - specifically, we need to be able to talk about base change for a bigger category of rings and not just the ones that are finitely presented over the base. Let us extend Definition 2.2 as follows:

Definition 2.4. *Let \mathcal{A} be an R -linear abelian category satisfying AB5. Using the fact that any R -module is the filtered colimit of finitely presented R -modules, we can extend definition 2.2 to the general case of*

$$- \otimes S : \mathcal{A} \rightarrow \mathcal{A}_S$$

for any R -algebra S .

The notion of base change category can be extended to the case of the derived category $D(\mathcal{A})$ of an abelian R -linear category \mathcal{A} in the obvious way:

Definition 2.5. *Given an inclusion of rings $R \hookrightarrow S$, the category $D(\mathcal{A})_S$ is given by pairs (C, ρ_C) where $C \in \text{Ob}(D(\mathcal{A}))$ and $\rho_C : S \rightarrow \text{Hom}_{D(\mathcal{A})}(C, C)$ is an R -algebra map such that the composition $R \rightarrow S \rightarrow \text{Hom}_{D(\mathcal{A})}(C, C)$ gives back the R -algebra structure on $D(\mathcal{A})$. The morphisms in $D(\mathcal{A})_S$ are the morphisms in $D(\mathcal{A})$ compatible with the S -structure.*

Again, we have a notion of tensor product:

Definition 2.6. *Let R be a ring, let \mathcal{A} be an R -linear abelian category satisfying AB5, and let C^\bullet be a complex of objects in \mathcal{A} :*

$$C^\bullet = \dots \rightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \rightarrow \dots$$

Let S be a ring, with a map $R \hookrightarrow S$. Then we can define $C^\bullet \otimes S$, as an object of $D(\mathcal{A}_S)$, as

$$C^\bullet \otimes S = \dots \rightarrow C^{i-1} \otimes S \xrightarrow{d^{i-1} \otimes 1} C^i \otimes S \xrightarrow{d^i \otimes 1} C^{i+1} \otimes S \rightarrow \dots$$

The complex $C^\bullet \otimes S$ can also be considered as an object of $D(\mathcal{A})_S$.

Remark 2.7. *Suppose that \mathcal{A} is a k -linear abelian category satisfying AB5 and $k \subset K$ is an extension of fields. In the situation of definition 2.4 and 2.6, similarly to the case of 2.3, it is easy to show that again tensoring with K is left adjoint to the forgetful functor*

- as a functor $\mathcal{A} \rightarrow \mathcal{A}_K$;
- as a functor $D(\mathcal{A}) \rightarrow D(\mathcal{A}_K)$;
- as a functor $D(\mathcal{A}) \rightarrow D(\mathcal{A})_K$.

Remark 2.8. *Let R be a ring, let \mathcal{A} be an R -linear abelian category satisfying AB5, and let C^\bullet be a complex of objects in \mathcal{A} ,*

$$C^\bullet = \dots \rightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \rightarrow \dots$$

Let $S \subset R$ a multiplicative system. In this case $C^\bullet \otimes_R S^{-1}R$, as an object of $D(\mathcal{A})$, is the same as

$$\dots \rightarrow \operatorname{colim}_{f \in S} f^{-1}C^{i-1} \xrightarrow{d^{i-1}} \operatorname{colim}_{f \in S} f^{-1}C^i \xrightarrow{d^i} \operatorname{colim}_{f \in S} f^{-1}C^{i+1} \rightarrow \dots$$

where $\operatorname{colim}_{f \in S} f^{-1}C^i$ is obtained by taking for every $f \in S$ a copy of C^i and as morphisms only the maps

$$f^{-1}C^i \longrightarrow (fg)^{-1}C^i$$

given by multiplication by $g : C^i \rightarrow C^i$.

Lemma 2.9. *In the situation of the remark above, if for every element $f \in S$ the multiplication by f is a quasi-isomorphism of C^\bullet , then the map*

$$C^\bullet \rightarrow C^\bullet \otimes_R S^{-1}R$$

is a quasi-isomorphism in $D(\mathcal{A})$.

Proof. Since taking cohomology commutes with directed colimits we have

$$H^i(C^\bullet \otimes_R S^{-1}R) = \operatorname{colim}_{f \in S} f^{-1}H^i(C^\bullet)$$

but since multiplication by any $g \in S$ is a quasi-isomorphism we get

$$f^{-1}H^i(C^\bullet) \xrightarrow[g]{\cong} (fg)^{-1}H^i(C^\bullet)$$

hence the cohomology of $C^\bullet \otimes_R S^{-1}R$ consists of only one copy of $H^i(C^\bullet)$, and the map $C^\bullet \rightarrow C^\bullet \otimes_R S^{-1}R$ is a quasi-isomorphism. \square

3. A RESULT ON BASE CHANGE FOR DERIVED CATEGORIES

The purpose of this section is to analyze the functor $D(\mathcal{A}_K) \rightarrow D(\mathcal{A})_K$ that sends an object in $D(\mathcal{A}_K)$ to the same object considered as an object of $D(\mathcal{A})$, together with its K -action. Specifically, we will prove the following:

Theorem 3.1. *Let \mathcal{A} be a k -linear abelian category satisfying AB5, where k is a field. Let $K = k(T)$ or $K = k(T, T')$. Then the functor*

$$\begin{aligned} D(\mathcal{A}_K) &\rightarrow D(\mathcal{A})_K \\ C^\bullet &\mapsto (C^\bullet, \rho_C) \end{aligned}$$

is essentially surjective, where $\rho_C : K \rightarrow \operatorname{Aut}(C^\bullet)$ is the obvious map.

Moreover, if L is a finite separable extension of $K = k(T)$ with $L = K(\alpha) = K[T]/P(T)$ then we can lift an object $(C^\bullet, \rho_C) \in D(\mathcal{A})_L$ to an object N^\bullet of $D(\mathcal{A}_K)$ endowed with a map $\psi_\alpha \in \operatorname{End}(N^\bullet)$ such that $P(\psi_\alpha)$ is zero on all cohomology groups, and the action of ψ_α on N^\bullet corresponds to the action of α on C^\bullet .

A stronger results for the case of a finite extension K/k was obtained in [Sos11]. In this case, there is actually an equivalence $D(\mathcal{A}_K) \rightarrow D(\mathcal{A})_K$.

The proof of this theorem will be carried out in several steps. First we will notice that, in the purely transcendental case $K = k(T, T')$, this comes down to lifting the actions of the two variables T and T' on a complex $C^\bullet \in D(\mathcal{A})_K$, given by $\rho_C(T)$ and $\rho_C(T')$, to actions coming from morphisms in \mathcal{A} that commute with each other.

Then in Lemma 3.3 we will tackle the case of one variable and obtain a complex $M^\bullet \in D(\mathcal{A}_{k[T]})$ with a quasi-isomorphism to C^\bullet as objects of $D(\mathcal{A})$, and such that the T -actions on M^\bullet and C^\bullet coincide. At this point, since $\rho_C(T)$ is an

automorphism of C^\bullet , tensoring with $k(T)$ will give us a complex in $D(\mathcal{A}_{k(T)})$ which is still quasi-isomorphic to C^\bullet .

A similar process can be repeated twice, as we will show in Lemma 3.4.

Lemma 3.2. *Let $e^n D(\mathcal{A})$ be the category whose*

- (1) *Objects are pairs $(C, \varphi_1, \dots, \varphi_n)$ where $E \in \text{Ob}(D(\mathcal{A}))$, $\varphi_i \in \text{End}_{D(\mathcal{A})}(C)$ for all i , and φ_i commutes with φ_j for all i, j ;*
- (2) *Morphisms $a : (C, \varphi_1, \dots, \varphi_n) \rightarrow (C', \varphi'_1, \dots, \varphi'_n)$ are elements $a \in \text{Hom}_{D(\mathcal{A})}(C, C')$ such that $a \circ \varphi_i = \varphi'_i \circ a$.*

Consider the full subcategory $e^n D'(\mathcal{A}) \subset e^n D(\mathcal{A})$ whose objects consist of those pairs $(C, \varphi_1, \dots, \varphi_n)$ such that for every nonzero $f \in k[T_1, \dots, T_n]$ the map $f(\varphi_1, \dots, \varphi_n) : C \rightarrow C$ is an isomorphism in $D(\mathcal{A})$.

The category $D(\mathcal{A})_{k(T_1, \dots, T_n)}$ is equivalent to the category $e^n D'(\mathcal{A})$. The equivalence is given by the functor

$$D(\mathcal{A})_{k(T_1, \dots, T_n)} \longrightarrow e^n D'(\mathcal{A}), \quad (C, \rho_C) \longmapsto (C, \rho_C(T_1), \dots, \rho_C(T_n)).$$

Proof. The equivalence is given by the inverse functor

$$e^n D'(\mathcal{A}) \longrightarrow D(\mathcal{A})_{k(T_1, \dots, T_n)} \\ (C, \varphi_1, \dots, \varphi_n) \mapsto \left(C, \begin{array}{ccc} \rho_C : k(T_1, \dots, T_n) & \rightarrow & \text{Aut} \\ T_i & \mapsto & \varphi_i \end{array} \right)$$

□

Lemma 3.3. *Let \mathcal{A} be a k -linear abelian category satisfying AB5, where k is a field. Let C^\bullet be a complex in $D(\mathcal{A})$. Let $\varphi \in \text{Hom}_{D(\mathcal{A})}(C^\bullet, C^\bullet)$. Then there exists a complex $M^\bullet \in D(\mathcal{A}_{k[T]})$ and a quasi-isomorphism $C^\bullet \rightarrow M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on M^\bullet corresponds to the action of multiplication by φ on C^\bullet .*

Note that when \mathcal{A} is a Grothendieck category, the same result can be achieved by considering the morphism corresponding to φ on a K -injective replacement of C^\bullet (which exists by [Fra01, Prop. 3.2]) and defining the action of T accordingly. However, in what follows we will need this specific form for the complex M^\bullet .

Proof. The map $\varphi : C^\bullet \rightarrow C^\bullet$ in $D(\mathcal{A})$ corresponds to a diagram of complexes in $D(\mathcal{A})$

$$\begin{array}{ccc} & Q^\bullet & \\ u \swarrow & & \searrow \varphi' \\ C^\bullet & \xrightarrow{\varphi} & C^\bullet \end{array}$$

where u is a quasi-isomorphism.

Let $C^\bullet[T] = C^\bullet \otimes_k k[T]$ as a complex in $D(\mathcal{A}_{k[T]})$. Consider the morphism $\varphi \otimes 1 - 1 \otimes T : C^\bullet[T] \rightarrow C^\bullet[T]$ in $D(\mathcal{A}_{k[T]})$. This can be represented by actual maps of complexes

$$\begin{array}{ccc} & Q^\bullet[T] & \\ u \otimes 1 \swarrow & & \searrow \varphi' \otimes 1 - u \otimes T \\ C^\bullet[T] & \xrightarrow{\varphi \otimes 1 - 1 \otimes T} & C^\bullet[T] \end{array}$$

The map $\varphi' \otimes 1 - u \otimes T$ is injective on all cohomology objects: to prove this we need to show that $\varphi' \otimes 1 - u \otimes T : H^r(Q^\bullet[T]) \rightarrow H^r(C^\bullet[T])$ is injective for every r .

Let $\alpha \in H^r(Q^\bullet[T])$, $\alpha \neq 0$, then

$$\alpha = \sum_{i=0}^n \alpha_i T^i$$

where all of the α_i are different from zero in $H^r(Q^\bullet)$. If

$$0 = (\varphi' \otimes 1 - u \otimes T)\alpha = \sum_{i=0}^n \varphi'(\alpha_i) T^i - \sum_{i=0}^n u(\alpha_i) T^{i+1}$$

then the only term of degree $n+1$ in T , $u(\alpha_n) T^{n+1}$, must be zero in $H^r(C^\bullet)$, hence $u(\alpha_n) = 0$, hence $\alpha_n = 0$ since u is a quasi-isomorphism. This contradicts our assumption that $\alpha_i \neq 0 \forall i$, and so this proves injectivity.

Now set

$$M^\bullet = \text{Cone}(Q^\bullet[T] \xrightarrow{\varphi' \otimes 1 - u \otimes T} C^\bullet[T])$$

Then we have a distinguished triangle

$$(1) \quad Q^\bullet[T] \xrightarrow{\varphi' \otimes 1 - u \otimes T} C^\bullet[T] \longrightarrow M^\bullet \longrightarrow (Q^\bullet[T])[1]$$

and by injectivity of the map $\varphi' \otimes 1 - u \otimes T$ on the cohomology objects we get a short exact sequence in cohomology

$$0 \rightarrow H^r(Q^\bullet[T]) \xrightarrow{\varphi' \otimes 1 - u \otimes T} H^r(C^\bullet[T]) \longrightarrow H^r(M^\bullet) \rightarrow 0$$

hence we get

$$H^r(M^\bullet) = \text{Coker}(H^r(Q^\bullet[T]) \xrightarrow{\varphi' \otimes 1 - u \otimes T} H^r(C^\bullet[T]))$$

for any r .

Now consider the composition

$$C^\bullet \longrightarrow C^\bullet[T] \xrightarrow{c} M^\bullet$$

This map is a quasi-isomorphism; to prove this we just need to show that under the map above, $H^r(C^\bullet) \cong \text{Coker}(H^r(Q^\bullet[T]) \xrightarrow{\varphi' \otimes 1 - u \otimes T} H^r(C^\bullet[T]))$ for every r .

Proceed as follows: first of all, considered as a sub-object of $H^r(C^\bullet[T])$ via the obvious map $C^\bullet \rightarrow C^\bullet[T]$, $H^r(C^\bullet)$ is not in the image of $\varphi' \otimes 1 - u \otimes T$, since, for any element $\alpha = \sum_{i=1}^n \alpha_i T^i$ of $H^r(Q^\bullet[T])$, its image $\sum_{i=1}^n \varphi'(\alpha_i) T^i - \sum_{i=0}^n u(\alpha_i) T^{i+1}$ is either zero or has a nonzero term of positive degree. To prove that any term of positive degree $\beta = \sum_{i=1}^n \beta_i T^i$ is in the image up to an element of degree zero, notice that it can be written as an element of lower degree plus an element of the image as follows:

$$\begin{aligned} \sum_{i=0}^n \beta_i T^i &= \sum_{i=0}^n \beta_i T^i - (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n) T^{n-1}) + (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n) T^{n-1}) = \\ &= \sum_{i=0}^n \beta_i T^i - \varphi'(u^{-1}(\beta_n)) T^{n-1} + \beta_n T^n + (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n) T^{n-1}) \\ &= \sum_{i=0}^{n-1} \beta_i T^i - \varphi'(u^{-1}(\beta_n)) T^{n-1} + (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n) T^{n-1}) \end{aligned}$$

Hence we found a complex $M^\bullet \in D(\mathcal{A}_{k[T]})$ which is quasi-isomorphic to C^\bullet as an object of $D(\mathcal{A})$; moreover the action of multiplication by φ on C^\bullet corresponds to the action by multiplication by T on M^\bullet , because the following diagram is commutative in $D(\mathcal{A})$:

$$\begin{array}{ccccc} C^\bullet & \longrightarrow & C^\bullet[T] & \xrightarrow{c} & M^\bullet \\ \downarrow \varphi & & \downarrow \varphi \otimes 1 & & \downarrow T \\ C^\bullet & \longrightarrow & C^\bullet[T] & \xrightarrow{c} & M^\bullet \end{array}$$

this follows from the fact that $c \circ (1 \otimes T) - (\varphi \otimes 1) \circ c = (1 \otimes T) \circ c - (\varphi \otimes 1) \circ c = (1 \otimes T - \varphi \otimes 1) \circ c = 0$ since those are two consecutive maps in a triangle. \square

Lemma 3.4. *Let \mathcal{A} be a k -linear abelian category satisfying AB5, where k is a field. Let C^\bullet be a complex in $D(\mathcal{A})$.*

Let $\varphi \in \text{Hom}_{D(\mathcal{A})}(C^\bullet, C^\bullet)$ such that $f(\varphi)$ is an isomorphism for all $f \in k[T]$ monic. Then there exists a complex $N^\bullet \in D(\mathcal{A}_{k(T)})$ and a quasi-isomorphism $C^\bullet \rightarrow N^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on N^\bullet corresponds to the action by multiplication by φ on C^\bullet .

Likewise, let $\psi \in \text{Hom}_{D(\mathcal{A})}(C^\bullet, C^\bullet)$ such that φ and ψ commute with each other and such that $f(\varphi, \psi)$ is a quasi-isomorphism for all $f \in k[T, T']$ nonzero. Then there exists a complex $N^\bullet \in D(\mathcal{A}_{k(T, T')})$ and a quasi-isomorphism $j : C^\bullet \rightarrow N^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T (resp. T') on N^\bullet corresponds to the action by multiplication by φ (resp. ψ) on C^\bullet .

Proof. By Lemma 3.3 we can find a complex $M^\bullet \in \mathcal{A}_{k[T]}$ and a quasi-isomorphism $j : C^\bullet \rightarrow M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on M^\bullet corresponds to the action by multiplication by φ on C^\bullet . This implies that multiplication by $f(T)$ gives a quasi-isomorphism of M^\bullet for all f monic.

Now let $N^\bullet := M^\bullet \otimes_{k[T]} k(T)$ as in Definition 2.6 above. This is a complex in $D(\mathcal{A}_{k(T)})$ and it is quasi-isomorphic to C^\bullet as objects of $D(\mathcal{A})$, by Lemma 2.9. The action of φ on C^\bullet corresponds to the action of T on N^\bullet .

For the second case, again by Lemma 3.3 we can find a complex $M^\bullet \in \mathcal{A}_{k[T]}$ and a quasi-isomorphism $j : C^\bullet \rightarrow M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on M^\bullet corresponds to the action by multiplication by φ on C^\bullet .

Moreover, we have an exact triangle

$$C^\bullet[T] \xrightarrow{\varphi \otimes 1 - 1 \otimes T} C^\bullet[T] \longrightarrow M^\bullet$$

in $D(\mathcal{A}_{k[T]})$, see (1).

Then, since φ and ψ commute with each other, we get a diagram in $D(\mathcal{A}_{k[T]})$:

$$\begin{array}{ccc} C^\bullet[T] & \xrightarrow{\varphi \otimes 1 - 1 \otimes T} & C^\bullet[T] \\ \psi \otimes 1 \downarrow & & \downarrow \psi \otimes 1 \\ C^\bullet[T] & \xrightarrow{\varphi \otimes 1 - 1 \otimes T} & C^\bullet[T] \end{array}$$

This diagram is commutative: this follows from the fact that $\varphi \circ \psi = \psi \circ \varphi$ in $D(\mathcal{A})$, hence $\varphi \psi \otimes 1 = \psi \varphi \otimes 1$ in $D(\mathcal{A}_{k[T]})$. Therefore we can find a map $\tilde{\psi}$ on M^\bullet so that

the following diagram commutes:

$$\begin{array}{ccccccc}
C^\bullet[T] & \xrightarrow{\varphi \otimes 1 - 1 \otimes T} & C^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (C^\bullet[T])[1] \\
\downarrow \psi \otimes 1 & & \downarrow \psi \otimes 1 & & \downarrow \tilde{\psi} & & \downarrow \psi \otimes 1 \\
C^\bullet[T] & \xrightarrow{\varphi \otimes 1 - 1 \otimes T} & C^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (C^\bullet[T])[1]
\end{array}$$

it follows that the action of $\tilde{\psi}$ on M^\bullet is the same as the action of ψ on C^\bullet , thanks to the commutativity of

$$\begin{array}{ccccc}
C^\bullet & \longrightarrow & C^\bullet[T] & \longrightarrow & M^\bullet \\
\downarrow \psi & & \downarrow \psi \otimes 1 & & \downarrow \tilde{\psi} \\
C^\bullet & \longrightarrow & C^\bullet[T] & \longrightarrow & M^\bullet
\end{array}$$

taking into account the fact that, as we mentioned already, the composition of the two horizontal maps is a quasi-isomorphism. As before we can then construct $P^\bullet = M^\bullet \otimes_{k[T]} k(T)$, which is quasi-isomorphic to M^\bullet and hence we get a corresponding map $\tilde{\psi} : P^\bullet \rightarrow P^\bullet$.

So we are in the following situation: we have a complex $P^\bullet \in D(\mathcal{A}_{k(T)})$ and a map $\tilde{\psi} : P^\bullet \rightarrow P^\bullet$ so that $f(\tilde{\psi})$ is a quasi-isomorphism for all $f \in k(T)[T']$ monic. By Lemma 3.3 again, we get a complex $Q^\bullet \in D((\mathcal{A}_{k(T)})_{k(T)[T']}) = D(\mathcal{A}_{k(T)[T']})$ which is quasi-isomorphic to P^\bullet .

Then define

$$N^\bullet := Q^\bullet \otimes_{k(T)[T']} k(T, T')$$

By Lemma 2.9, since $f(T, \psi)$ is a quasi-isomorphisms for all nonzero $f \in k(T)[T']$, the complex $N^\bullet \in D(\mathcal{A}_{k(T, T')})$ is quasi isomorphic to Q^\bullet as objects of $D(\mathcal{A}_{k(T)[T']})$ hence it is quasi-isomorphic to C^\bullet as objects of $D(\mathcal{A})$. The action of φ and ψ correspond to the action of T and T' respectively. \square

The last thing we need to do is to tackle the case of a general separable field extension of transcendence degree one, corresponding to the last statement of Theorem 3.1:

Lemma 3.5. *Let \mathcal{A} be a k -linear abelian category satisfying AB5, where k is a field. Let C^\bullet be a complex in $D(\mathcal{A})$. Let $\varphi, \psi \in \text{Hom}_{D(\mathcal{A})}(C^\bullet, C^\bullet)$ such that φ and ψ commute with each other, and such that $f(\varphi)$ is a quasi-isomorphisms for all $f \in k[T]$ monic and there exists an irreducible $P \in k[T, T']$ with $P(\varphi, \psi) = 0$.*

Then there exists a complex $N^\bullet \in D(\mathcal{A}_{k(T)})$ and a quasi-isomorphism $j : C^\bullet \rightarrow N^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on N^\bullet corresponds to the action by multiplication by φ on C^\bullet . Moreover there is a morphism $\tilde{\psi} \in \text{End}(N^\bullet)$ such that the action of ψ on C^\bullet corresponds to the action of $\tilde{\psi}$ on N^\bullet and $P(T, \tilde{\psi})$ induces the zero map on all cohomology groups of N^\bullet .

Proof. By Lemma 3.3 we can find a complex $M^\bullet \in \mathcal{A}_{k[T]}$ and a quasi-isomorphism $j : C^\bullet \rightarrow M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on M^\bullet corresponds to the action by multiplication by φ on C^\bullet .

Moreover, we have an exact triangle

$$C^\bullet[T] \xrightarrow{\varphi \otimes 1 - 1 \otimes T} C^\bullet[T] \longrightarrow M^\bullet$$

in $D(\mathcal{A}_{k[T]})$, see (1).

Then, since φ and ψ commute with each other, we get a commutative diagram in $D(\mathcal{A}_{k[T]})$

$$\begin{array}{ccc} C^\bullet[T] & \xrightarrow{\varphi \otimes 1 - 1 \otimes T} & C^\bullet[T] \\ \psi \otimes 1 \downarrow & & \downarrow \psi \otimes 1 \\ C^\bullet[T] & \xrightarrow{\varphi \otimes 1 - 1 \otimes T} & C^\bullet[T] \end{array}$$

Therefore we can find a map $\tilde{\psi}$ on M^\bullet so that the following diagram commutes:

$$\begin{array}{ccccccc} C^\bullet[T] & \xrightarrow{\varphi \otimes 1 - 1 \otimes T} & C^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (C^\bullet[T])[1] \\ \psi \otimes 1 \downarrow & & \downarrow \psi \otimes 1 & & \downarrow \tilde{\psi} & & \downarrow \psi \otimes 1 \\ C^\bullet[T] & \xrightarrow{\varphi \otimes 1 - 1 \otimes T} & C^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (C^\bullet[T])[1] \end{array}$$

Since $P(T, \psi) = 0$, we obtain that $P(\varphi \otimes 1, \psi \otimes 1) = 0$ in $D(\mathcal{A}_{k[T]})$, hence $P(T, \psi \otimes 1)$ is zero on $C[T]$.

As before we can construct $N^\bullet = M^\bullet \otimes_{k[T]} k(T)$, which is quasi-isomorphic to M^\bullet and hence we get a corresponding map $\tilde{\psi} : N^\bullet \rightarrow N^\bullet$ and the action of ψ on C^\bullet corresponds to the action of $\tilde{\psi}$ on N^\bullet .

Finally, since $P(T, \psi \otimes 1)$ is zero on $C[T]$, it follows that $P(T, \tilde{\psi}) = 0$ induces the zero map on all cohomology of M^\bullet and hence of N^\bullet . \square

We are now ready to prove Theorem 3.1:

Proof of Theorem 3.1. By Lemma 3.2, we just need to show that the functors

$$\begin{aligned} D(\mathcal{A}_K) &\rightarrow e^1 D'(\mathcal{A}) \\ C^\bullet &\mapsto (C^\bullet, \cdot T) \end{aligned}$$

and

$$\begin{aligned} D(\mathcal{A}_K) &\rightarrow e^2 D'(\mathcal{A}) \\ C^\bullet &\mapsto (C^\bullet, \cdot T, \cdot T') \end{aligned}$$

are essentially surjective.

Let $(E, \varphi) \in e^1 D'(\mathcal{A})$. Then by Lemma 3.4 there exists $N^\bullet \in \mathcal{A}_{k(T)}$ such that N is quasi isomorphic to E and the action of φ on E^\bullet corresponds to the action of T on N^\bullet . This proves the case $i = 1$.

Similarly, let $(E, \varphi, \varphi') \in e^2 D'(\mathcal{A})$. Then by Lemma 3.4 there exists $N^\bullet \in \mathcal{A}_{k(T, T')}$ such that N is quasi isomorphic to E and the action of φ and φ' on E^\bullet correspond to the action of T and T' respectively on N^\bullet . This proves the case $i = 2$.

The last part follows from Lemma 3.5 by setting $\psi_\alpha := \tilde{\psi}$. \square

Let us now apply this theorem to the case $\mathcal{A} = \text{QCoh}(X)$, where X is a quasi-compact, separated scheme over a field k . This is possible since $\text{QCoh}(X)$ satisfies AB5. Moreover, note that in this case we have an equivalence $D_{\text{Qcoh}}(X) \cong D(\text{Qcoh}(X))$. As a preliminary step, we will prove the following technical lemma:

Lemma 3.6. *Let $k \subset K$ be a field extension, X a quasi-compact and separated scheme. Let $X_K \xrightarrow{j} X$ the base change morphism. Then there is an equivalence of categories*

$$D_{\mathrm{QCoh}}(X_K) \xrightarrow{\Psi} D(\mathrm{QCoh}(X)_K)$$

under this equivalence, the functors

$$Lj^*, \cdot \otimes K : D_{\mathrm{QCoh}}(X) \rightarrow D(\mathrm{QCoh}(X_K))$$

and

$$Rj_*, \mathrm{Forget} : D_{\mathrm{QCoh}}(X_K) \rightarrow D(\mathrm{QCoh}(X))$$

coincide.

In other words,

$$Rj_* = \mathrm{Forget} \circ \Psi : D(\mathrm{QCoh}(X)_K) \rightarrow D_{\mathrm{QCoh}}(X)$$

$$\Psi \circ Lj^* = - \otimes K : D_{\mathrm{QCoh}}(X) \rightarrow (D_{\mathrm{QCoh}}(X))_K$$

This is summarized in the following diagram:

$$\begin{array}{ccc}
 & D_{\mathrm{QCoh}}(X) & \\
 j_* \swarrow & \begin{array}{c} \text{Forget} \uparrow \\ \downarrow \otimes K \end{array} & \searrow Lj^* \\
 & (D_{\mathrm{QCoh}}(X))_K & \\
 & \uparrow \Psi & \\
 D_{\mathrm{QCoh}}(X_K) = D(\mathrm{QCoh}(X_K)) & &
 \end{array}$$

Proof. There is an equivalence of categories induced by j_* between quasi-coherent \mathcal{O}_{X_K} -modules and quasi-coherent $j_*\mathcal{O}_{X_K}$ -modules on X . But $j_*\mathcal{O}_{X_K} = \mathcal{O}_X \otimes K$ and an $(\mathcal{O}_X \otimes K)$ -module is the same thing as an \mathcal{O}_X -module with a K -structure which is compatible with its k -structure.

Hence we get an equivalence

$$\begin{aligned}
 \Psi : \mathrm{QCoh}(X_K) &\rightarrow \mathrm{QCoh}(X)_K \\
 C &\mapsto (j_*C, \rho_C)
 \end{aligned}$$

where ρ_C is the composition $K \rightarrow \mathcal{O}_X \otimes K \rightarrow \mathrm{End}(j_*C)$.

Under this equivalence, the two functors j_* and “Forget” coincide; moreover, always under the same equivalence, both j^* and $- \otimes K$ are left adjoint to j_* , hence they also coincide.

Thus all of this also holds for the corresponding derived categories; hence the statement follows since $D_{\mathrm{QCoh}}(X) = D(\mathrm{QCoh}(X))$ for X quasi compact and separated. \square

Corollary 3.7. *Let X be a quasi compact, separated scheme over a field k .*

Let $K = k(T)$ or $K = k(T, T')$.

The map

$$\begin{aligned}
 D_{\mathrm{QCoh}}(X_K) &\longrightarrow (D_{\mathrm{QCoh}}(X))_K \\
 C^\bullet &\mapsto (\mathrm{Forget}(C^\bullet), \rho_C)
 \end{aligned}$$

is essentially surjective, where ρ_C is the obvious K -structure on C .

Moreover, if L is a finite separable extension of $K = k(T)$ with $L = K(\alpha) = K[T]/P(T)$ then we can lift an object $(C^\bullet, \rho_C) \in (D_{\mathrm{QCoh}}(X))_L$ to an object N^\bullet of

$D_{\text{QCoh}}(X_K)$ endowed with a map $\tilde{\psi} \in \text{End}(N^\bullet)$ such that $P(\tilde{\psi})$ induces the zero map on all cohomology groups of N^\bullet .

Proof. By Lemma 3.6, there is an equivalence between $D_{\text{QCoh}}(X_K)$ and $D(\text{QCoh}(X)_K)$, hence it is sufficient to show that the map

$$\begin{aligned} D(\text{QCoh}(X)_K) &\rightarrow (D(\text{QCoh}(X)))_K \\ C^\bullet &\mapsto (\text{Forget}(C^\bullet), \rho_C) \end{aligned}$$

is essentially surjective.

Let $\mathcal{A} = \text{QCoh}(X)$. This category satisfies AB5, hence theorem 3.1 applies in this case. \square

4. A REPRESENTABILITY THEOREM FOR DERIVED CATEGORIES

The results of the previous section will become handy to study functors from $D_{\text{Coh}}^b(X)$, where X is defined over a field k , to a vector space over a bigger field in light of the following theorem:

Theorem 4.1. *Let k be a field, \mathcal{A} be a k -linear abelian category satisfying AB5, and let $k \hookrightarrow K$ an inclusion of fields. Let $D(\mathcal{A})^c$ denote the full subcategory of compact objects in $D(\mathcal{A})$.*

Given an exact, contravariant functor

$$F : D(\mathcal{A})^c \rightarrow \underline{\text{mod}}_K$$

there exists a $T \in D(\mathcal{A})_K$ such that

$$F(C) = \text{Mor}_{D(\mathcal{A})_K}(C \otimes K, T)$$

for all $C \in D(\mathcal{A})^c$.

To prove this we will use the ideas from [CKN01, Lemma 2.14] where the version of this theorem with $k = K$ has been proved for a general triangulated category.

Proof of theorem 4.1. Let D be the functor taking a K -vector space to its dual. Then $G = D \circ F$ is exact and covariant. Let \mathcal{T} be the cocomplete triangulated subcategory generated by $D(\mathcal{A})^c$, i.e. the smallest full triangulated subcategory of $D(\mathcal{A})$ containing $D(\mathcal{A})^c$ which is closed under colimits.

Let $\tilde{G} : \mathcal{T} \rightarrow \underline{\text{Mod}}_K$ be the Kan extension of G to \mathcal{T} : this is defined as

$$\tilde{G}(C) = \text{colim}_{\substack{B \rightarrow C \\ B \in D(\mathcal{A})^c}} G(B)$$

Since \tilde{G} is exact and commutes with coproducts, it follows that $D \circ \tilde{G}$ is exact and takes coproducts to products. Hence by the Brown representability theorem [Nee01, Theorem 8.3.3] the functor $D \circ \tilde{G}$ is representable, as a functor to $\underline{\text{Mod}}_k$, by an object $U \in \mathcal{T} \subset D(\mathcal{A})$.

The K -action on $\underline{\text{Mod}}_K$ induces a K -action $\tilde{\rho}$ on $D \circ \tilde{G} = h_U$, hence by Yoneda we get a K -action ρ on U , given by $K \xrightarrow{\rho} \text{Nat}(h_U, h_U) = \text{Aut}(U)$. Therefore we obtain an object $(U, \rho) \in D(\mathcal{A})_K$. We need to show that

$$D \circ \tilde{G}(C) = \text{Mor}_{D(\mathcal{A})_K}(C \otimes K, (U, \rho))$$

for all $C \in D(\mathcal{A})^c$.

To do so, first of all notice that as k -vector spaces

$$D \circ \tilde{G}(C) = \text{Mor}_{D(\mathcal{A})}(C, U) = \text{Mor}_{D(\mathcal{A})_K}(C \otimes K, (U, \rho))$$

because $K \otimes_k -$ is left adjoint to the functor forgetting the K -structure. By our definition of the K -action on $\text{Mor}_{D(\mathcal{A})}(C, U)$, this is the same as the K -action on $D \circ \tilde{G}(C)$; moreover the k -vector space map

$$\begin{aligned} \text{Mor}_{D(\mathcal{A})}(C, U) &\xrightarrow{\gamma} \text{Mor}_{D(\mathcal{A})_K}(C \otimes K, (U, \rho)) \\ f &\mapsto f \otimes \rho \end{aligned}$$

is compatible with the K -action since, for any $\alpha \in K$,

$$\gamma(\alpha \cdot f) = \gamma(\tilde{\rho}(\alpha)f) = \tilde{\rho}(\alpha)f \otimes \rho(\cdot) = f \otimes \rho(\alpha)\rho(\cdot) = \alpha \cdot (f \otimes \rho(\cdot))$$

hence we found that the two actions coincide and so

$$D \circ \tilde{G}(C) = \text{Mor}_{D(\mathcal{A})_K}(C \otimes K, (U, \rho))$$

Let $T = (U, \rho)$. Now since F is of finite type, we get

$$F(C) = (D \circ D \circ F)(C) = (D \circ G)(C) = (D \circ \tilde{G})(C) = \text{Mor}_{D(\mathcal{A})_K}(C \otimes K, T)$$

□

Lemma 4.2. *Let k and K be two fields, $k \hookrightarrow K$.*

Consider the equivalence of categories

$$D^b(\underline{\text{mod}}(\Lambda)) \xrightarrow{\theta} D^b(\text{Coh}(\mathbb{P}_k^n))$$

as described in [Bei78].

Then there is also an equivalence of categories

$$D^b(\underline{\text{mod}}(\Lambda \otimes K)) \xrightarrow{\theta_K} D^b(\text{Coh}(\mathbb{P}_K^n))$$

and the diagram

$$\begin{array}{ccc} D^b(\underline{\text{mod}}(\Lambda)) & \xrightarrow{\theta} & D^b(\text{Coh}(\mathbb{P}_k^n)) \\ \downarrow & & \downarrow \\ D^b(\underline{\text{mod}}(\Lambda \otimes K)) & \xrightarrow{\theta_K} & D^b(\text{Coh}(\mathbb{P}_K^n)) \end{array}$$

is commutative.

Proof. By [Bei78], we have $\Lambda = \text{End}(\mathcal{M})$ where $\mathcal{M} = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}_k^n}(i)$. Set $\mathcal{M}_K = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}_K^n}(i)$, then

$$\begin{aligned} \text{End}_{\mathbb{P}_K^n}(\mathcal{M}_K) &= \text{End}_{\mathbb{P}_K^n} \left(\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}_K^n}(i) \right) = \bigoplus_{i,j=0}^n \text{End}_{\mathbb{P}_K^n}(\mathcal{O}_{\mathbb{P}_K^n}(i), \mathcal{O}_{\mathbb{P}_K^n}(j)) = \\ &= \bigoplus_{i,j=0}^n K[x_0, \dots, x_n]_{j-i} = \bigoplus_{i,j=0}^n k[x_0, \dots, x_n]_{j-i} \otimes K \\ &= \left(\bigoplus_{i,j=0}^n k[x_0, \dots, x_n]_{j-i} \right) \otimes K = \Lambda \otimes K \end{aligned}$$

Moreover, the equivalence θ is induced by the map

$$\underline{\text{mod}}(\Lambda) \xrightarrow{- \otimes_{\Lambda} \mathcal{M}} \text{Coh}(\mathbb{P}_k^n)$$

and if we let $h : \mathbb{P}_K^n \rightarrow \mathbb{P}_k^n$ be the base change morphism, we obtain the following commutative diagram:

$$\begin{array}{ccc} \underline{\text{mod}}(\Lambda) & \xrightarrow{-\otimes_{\Lambda} \mathcal{M}} & \text{Coh}(\mathbb{P}_k^n) \\ \otimes K \downarrow & & \downarrow h^* \\ \underline{\text{mod}}(\Lambda \otimes K) & \xrightarrow{-\otimes_{\Lambda \otimes K} \mathcal{M}_{\kappa}} & \text{Coh}(\mathbb{P}_K^n) \end{array}$$

this proves the last assertion. \square

We are now almost ready to prove Theorem 1.1, but first we will prove the version of the theorem for the purely transcendental case. The following proof uses ideas from [BVdB03, Theorem A.1].

Theorem 4.3. *Let X be a smooth projective variety over a field k . Let $K = k(T)$ or $K = k(T, T')$. Consider a contravariant, cohomological, finite type functor*

$$H : D_{\text{Coh}}^b(X) \rightarrow \underline{\text{mod}}_K$$

Then the complex T of Theorem 4.1 lifts to a complex $S \in D_{\text{Coh}}^b(X_K)$ such that H is representable by S , i.e. for every $C \in D_{\text{Coh}}^b(X)$ we have

$$H(C) = \text{Mor}_{D_{\text{Coh}}^b(X_K)}(Lj^*C, S)$$

where $j : X_K \rightarrow X$ is the base change morphism.

Proof. By Lemma 4.1, the functor H is representable by an element $T \in (D_{\text{QCoh}}(X))_K$, i.e.

$$H(C) = \text{Mor}_{(D_{\text{QCoh}}(X))_K}(C \otimes K, T)$$

Let S be a lift of T to $D_{\text{QCoh}}(X_K)$ (this is possible by Corollary 3.7). Let C be an element of $D_{\text{Coh}}^b(X)$. By applying the functors in Lemmas 3.1 and 3.6 we get a K -linear map

$$\text{Mor}_{D_{\text{QCoh}}(X_K)}(Lj^*C, S) \xrightarrow{\Psi(\cdot)} \text{Mor}_{(D_{\text{QCoh}}(X))_K}(\Psi \circ Lj^*C, T)$$

and, since by Lemma 3.6, $\Psi \circ Lj^*C = C \otimes K$, we have

$$\text{Mor}_{(D_{\text{QCoh}}(X))_K}(\Psi \circ Lj^*C, T) = \text{Mor}_{(D_{\text{QCoh}}(X))_K}(C \otimes K, T) = H(C)$$

Hence to show that H is represented by S we just need to show that $\Psi(\cdot)$ is an isomorphism. It suffices to show that it is an isomorphism of k -vector spaces, which follows from the following diagram of k -vector spaces:

$$\begin{array}{ccc} \text{Mor}_{D_{\text{QCoh}}(X_K)}(Lj^*C, S) & \xrightarrow{\Psi(\cdot)} & \text{Mor}_{(D_{\text{QCoh}}(X))_K}(\Psi \circ Lj^*C, T) \\ \parallel & & \parallel \\ \text{Mor}_{D_{\text{QCoh}}(X)}(C, Rj_*S) & & \text{Mor}_{(D_{\text{QCoh}}(X))_K}(C \otimes K, T) \\ \parallel & & \parallel \\ \text{Mor}_{D_{\text{QCoh}}(X)}(C, \text{Forget}(\Psi(S))) & = & \text{Mor}_{(D_{\text{QCoh}}(X))_K}(C, \text{Forget}(T)) \end{array}$$

here we used the fact that $Rj_* = \text{Forget} \circ \Psi$, again from Lemma 3.6.

So $\Psi(\cdot)$ is an isomorphism, and hence H is represented by $S \in D_{\text{QCoh}}(X_K)$. We still have to show that S is actually in $D_{\text{Coh}}^b(X_K)$.

Choose an embedding $\pi : X \rightarrow \mathbb{P}_k^n$. Let $H' = H \circ L\pi^*$. Let $\theta : D^b(\underline{\text{mod}}(\Lambda)) \rightarrow D^b(\text{Coh}(\mathbb{P}_k^n))$ and $\theta_K : D^b(\underline{\text{mod}}(\Lambda \otimes K)) \rightarrow D^b(\text{Coh}(\mathbb{P}_K^n))$ as defined in Lemma 4.2 above. Let $H'' = H' \circ \theta$. Let $h : \mathbb{P}_K^n \rightarrow \mathbb{P}_k^n$ be the base change morphism.

Consider the following diagram:

$$\begin{array}{ccccccc}
 & & & & H'' & & \\
 & & & & \curvearrowright & & \\
 & & & & H' & & \\
 & & & & \curvearrowright & & \\
 D^b(\underline{\text{mod}}(\Lambda)) & \xrightarrow{\theta} & D^b(\text{Coh}(\mathbb{P}_k^n)) & \xrightarrow{L\pi^*} & D_{\text{Coh}}^b(X) & \xrightarrow{H} & \underline{Vect}_K \\
 \downarrow -\otimes K & & \downarrow h^* & & \downarrow Lj^* & & \\
 D^b(\underline{\text{mod}}(\Lambda \otimes K)) & \xrightarrow{\theta_K} & D^b(\text{Coh}(\mathbb{P}_K^n)) & \xrightarrow{L\pi_K^*} & D_{\text{Coh}}^b(X_K) & &
 \end{array}$$

and let $C \in D^b(\text{Coh}(\mathbb{P}_k^n))$.

$$\begin{aligned}
 H'(C) &= H(L\pi^*(C)) = \text{Mor}_{D_{\text{QCoh}}(X_K)}(Lj^*L\pi^*C, S) = \\
 &= \text{Mor}_{D_{\text{QCoh}}(X_K)}(L\pi_K^*h^*C, S) = \text{Mor}_{D_{\text{QCoh}}(\mathbb{P}_K^n)}(h^*C, R\pi_{K*}S)
 \end{aligned}$$

so H' is represented by $R\pi_{K*}S \in D_{\text{QCoh}}(\mathbb{P}_K^n)$.

Let $V = \theta_K^{-1}(R\pi_{K*}(S))$ so that H'' is represented by V . Then

$$H''(\Lambda) = \text{Mor}_{\Lambda \otimes K}(\Lambda \otimes K, V)$$

and

$$\begin{aligned}
 \sum_n \dim H''(\Lambda[n]) &= \sum_n \dim \text{Mor}(\Lambda[n] \otimes K, V) = \\
 &= \sum_n \dim \text{Mor}((\Lambda \otimes K)[n], V) < \infty
 \end{aligned}$$

since H'' is of finite type. Therefore $V \in D^b(\underline{\text{mod}}(\Lambda \otimes K))$.

This implies that $R\pi_{K*}S \in D^b(\text{Coh}(\mathbb{P}_K^n))$ hence $S \in D^b(\text{Coh}(X_K))$. \square

Proof of theorem 1.1. The case where L is purely transcendental of degree 2 over k was treated in Theorem 4.3. Let L be a finitely generated separable field extension of k with $\text{trdeg}_k L \leq 1$. There exists a field K such that K is a purely transcendental extension of k of degree less than or equal 1, and $K \subset L$ is a finite extension. Set $L = K(\alpha) = K[T]/P(T)$, where $P(T)$ is a separable polynomial. Consider the composition

$$\begin{array}{ccccc}
 & & H' & & \\
 & & \curvearrowright & & \\
 D_{\text{Coh}}^b(X) & \xrightarrow{H} & \underline{\text{mod}}_L & \xrightarrow{\text{Forget}} & \underline{\text{mod}}_K
 \end{array}$$

By theorem 4.3, H' is representable by an object $S \in D_{\text{Coh}}^b(X_K)$. Moreover, by Corollary 3.7, S is endowed with a map ψ_α such that $P(\psi_\alpha) = 0$ is zero on all the cohomology groups of S .

First of all, this implies that there exists an n such that $P(\psi_\alpha)^n = 0$. In fact, considering the good truncations $\tau_{\leq i}S$,

$$\dots \rightarrow S^{i-1} \rightarrow Z^i \rightarrow 0,$$

the claim follows inductively considering the distinguished triangles

$$\tau_{\leq i-1}S \rightarrow \tau_{\leq i}S \rightarrow H^i(S)[-i]$$

and recalling the fact that if (f_1, f_2, f_3) is a morphism between two triangles and two of the morphisms are nilpotent, then so is the third.

Now let $h : X_L \rightarrow X_K$ be the base change morphism, and consider the pullback $Lh^*S \in D_{\text{Coh}}^b(X_L)$. It has an $L[T]$ action induced by the morphism $Lh^*\psi_\alpha$, and $P(Lh^*\psi_\alpha)^n = 0$ so Lh^*S has in fact an $L[T]/P^n(T)$ -action. But since P is a separable polynomial, the map $L[T]/P^n(T) \rightarrow L[T]/P(T)$ splits as L -algebra map hence $L[T]/P(T)$ also acts on Lh^*S .

Since, over L , $P(T)$ factors as $(T - \alpha)Q(T)$, we can find two elements e_1, e_2 of $L[T]/P(T)$ such that $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1e_2 = 0$, $e_1 + e_2 = 1$. But since $L[T]/P(T)$ acts on Lh^*S , this gives two idempotent operators e_1, e_2 in $\text{Aut}_{D_{\text{Coh}}^b(X_L)}(Lh^*S)$ such that $e_1e_2 = 0$, $e_1 + e_2 = \text{id}_{Lh^*S}$.

Now since $D_{\text{Coh}}^b(X_L)$ is Karoubian by [BN93, Proposition 3.2] we have obtained that $Lh^*S = E \oplus S_2$ and $Lh^*\psi_\alpha$ acts as multiplication by α on E .

We claim that $Rh_*E = S$. Consider the map

$$S \rightarrow Rh_*Lh^*S \xrightarrow{pr_1} Rh_*E$$

Under the identification $D_{\text{QCoh}}(X_L) \xrightarrow{\Psi} D(\text{QCoh}(X)_L)$ this corresponds to $S \rightarrow S \otimes L \rightarrow \text{Forget}(E)$, so this is actually the identity map on S .

Then for every $C \in D_{\text{Coh}}^b(X)$ we have a map of L -vector spaces

$$\text{Mor}_{D_{\text{QCoh}}(X_L)}(Lj^*C, E) \rightarrow \text{Mor}_{(D_{\text{QCoh}}(X))_L}(C \otimes L, T) = H(C)$$

where $j : X_L \rightarrow X$ is the base change morphism, since E is a lift of S to $D_{\text{Coh}}^b(X_L)$ with the correct L -action. This map is an isomorphism because it is an isomorphism of K -vector spaces:

$$\begin{aligned} \text{Mor}_{D_{\text{QCoh}}(X_L)}(Lj^*C, E) &= \text{Mor}_{(D_{\text{QCoh}}(X_K))}(Li^*C, Rh_*E) \\ &= \text{Mor}_{(D_{\text{QCoh}}(X_K))}(Li^*C, S) = H'(C) \end{aligned}$$

where $i : X_K \rightarrow X$ is the base change morphism. □

Proof of theorem 1.2. Consider the composition

$$\begin{array}{ccccccc} D_{\text{Coh}}^b(X) & \xrightarrow{F} & D_{\text{Coh}}^b(Y) & \xrightarrow{i^*} & D_{\text{Coh}}^b(\eta) & \xrightarrow{H^0} & \underline{\text{mod}}_{K(Y)} \xrightarrow{D} \underline{\text{mod}}_{K(Y)} \\ & & & & & \searrow & \nearrow \\ & & & & & H & \end{array}$$

where $H^0(-) = H^0(\eta, -)$ and D is the dual as $K(Y)$ -vector space. H is an exact contravariant finite type functor, hence by theorem 1.1 it is representable by $E \in D_{\text{Coh}}^b(X_{K(Y)})$.

Now consider the following diagram:

$$\begin{array}{ccc}
 X_{K(Y)} & \xrightarrow{p} & \eta \\
 \downarrow g & & \downarrow i \\
 X \times Y & \xrightarrow{\pi_2} & Y \\
 \downarrow \pi_1 & & \downarrow \\
 X & \longrightarrow & \text{Spec } k
 \end{array}
 \quad \begin{array}{l}
 \\
 j \curvearrowright \\
 \\
 \end{array}$$

Both squares are cartesian because $X_{K(Y)} = X \times_{\text{Spec } k} K(Y) = X \times Y \times_Y \eta$. Note that g is a flat map, so the derived pullback is just regular pullback in every degree. Also, g is an affine map so that pushforward is also exact.

Let $E^\vee = \underline{\text{RHom}}_{X_{K(Y)}}(E, \mathcal{O}_{X_{K(Y)}})$. Let us construct a complex $A \in D_{\text{Coh}}^b(X \times Y)$ such that $Lg^*A = E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}]$.

Let $\mathcal{L} \in \text{Coh}(X \times Y)$ be a line bundle such that $E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes g^*\mathcal{L}^{\otimes n} \in D_{\text{Coh}}^b(X_{K(Y)})$ is generated by its global sections in each degree. Let $\{s_{i,\ell}\}$ be a set of generators in degree ℓ . Consider the complex $g_*(E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes g^*\mathcal{L}^{\otimes n})$ on $D_{\text{Coh}}^b(X \times Y)$. Then take the subcomplex generated in each degree by $\{g_*s_{i,\ell}\} \cup \{g_*ds_{i,\ell-1}\}$, and twist it down by \mathcal{L}^{-n} . This gives the desired complex $A \in D_{\text{Coh}}^b(X \times Y)$.

Then we get the following:

$$\begin{aligned}
 H^0 \circ i^* \circ \Phi_A(C) &= H^0 i^* R\pi_{2*}(A \otimes \pi_1^* C) \\
 &= H^0 R p_*(g^* A \otimes g^* \pi_1^* C) \quad (\text{by flat base change}) \\
 &= H^0 R p_*(E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C) \\
 &= \text{Mor}(\mathcal{O}_\eta, R p_*(E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C)) \\
 &= \text{Mor}(p^* \mathcal{O}_\eta, E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C) \\
 &= \text{Mor}(\mathcal{O}_{X_{K(Y)}}, E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C) \\
 &= \text{Mor}(E, \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C) \\
 &= D \circ \text{Mor}(j^* C, E) \\
 &= D \circ H(C) \\
 &= H^0 \circ i^* \circ F(C)
 \end{aligned}$$

for every $C \in D_{\text{Coh}}^b(X)$.

Now since F is an exact functor,

$$\begin{aligned}
 H^i \circ i^* \circ F &= H^0(i^* \circ F(C)[i]) = H^0(i^* \circ F(C[i])) = \\
 &= H^0(i^* \circ \Phi_A(C[i])) = H^0(i^* \circ \Phi_A(C)[i]) = \\
 &= H^i \circ i^* \circ \Phi_A(C)
 \end{aligned}$$

Hence, since all cohomology groups agree and $D_{\text{Coh}}^b(K(Y))$ is equivalent to the category of graded vector spaces over $K(Y)$, F and Φ_A agree after restricting to the generic point of Y . \square

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